Tan’s contact for one-dimensional SU(N) Fermi gases

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$$:

Multicomponent 1D fermions with repulsive interactions: a new system for studying....

- Effects of strong interactions and correlations

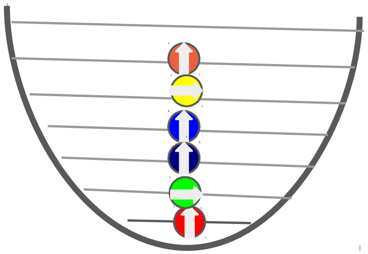
- Universality

- Beyond-Luttinger-liquid phenomena

- Magnetic phases : analog of antiferromagnetism, itinerant ferromagnetism

Plan

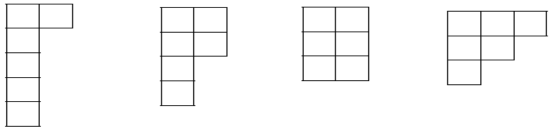
1D multicomponent fermions with repulsive interactions :



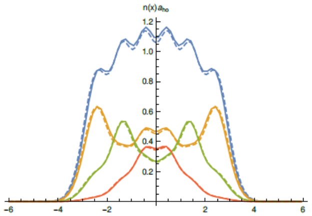
Exact solution at infinite interactions

Numerical results at arbitrary interactions

– Symmetry characterization of the wavefunction

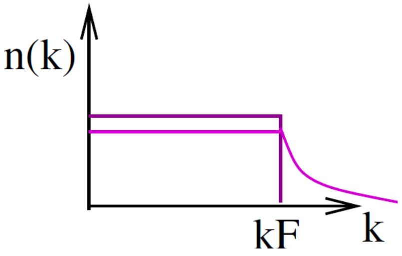


– Density profiles



– Momentum distribution

and Tan's contact



1D two-component Fermi gases

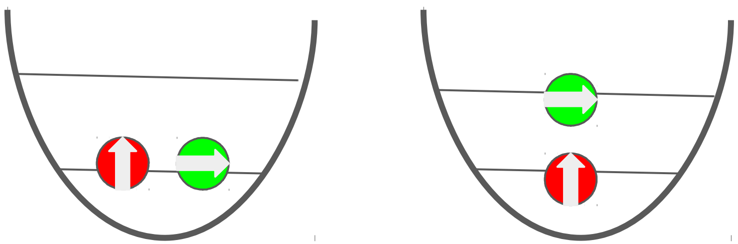
with repulsive intercomponent interactions ;
like electrons with spin 1/2

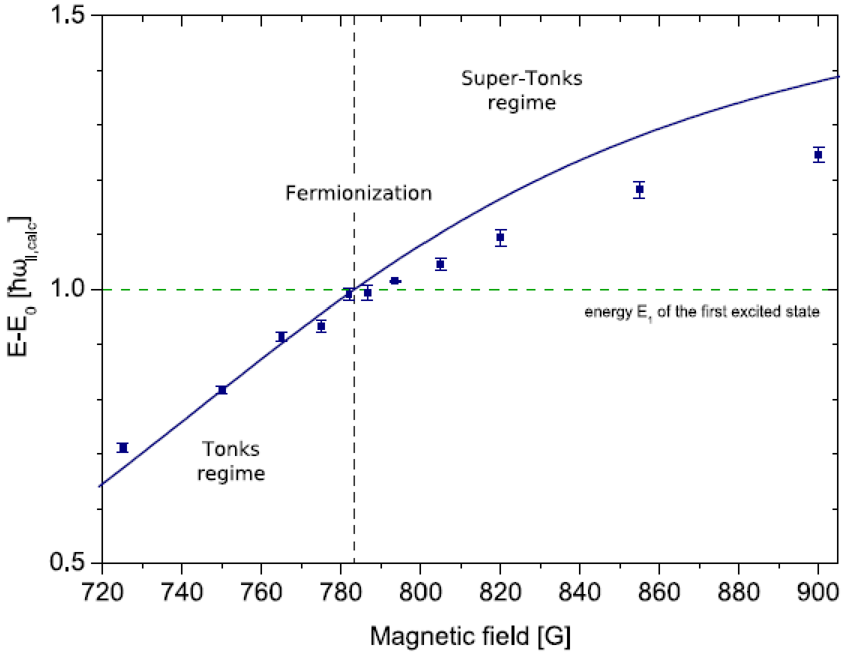
Tuning the interactions : possibility to reach strongly correlated regime

Fermionizing the fermions:

strong repulsive interactions → effective Pauli principle between fermions belonging to different components → 'Tonks-Girardeau regime'

at increasing interactions….





[Zurn et al, Phys Rev Lett 108, 070503 (2012)]

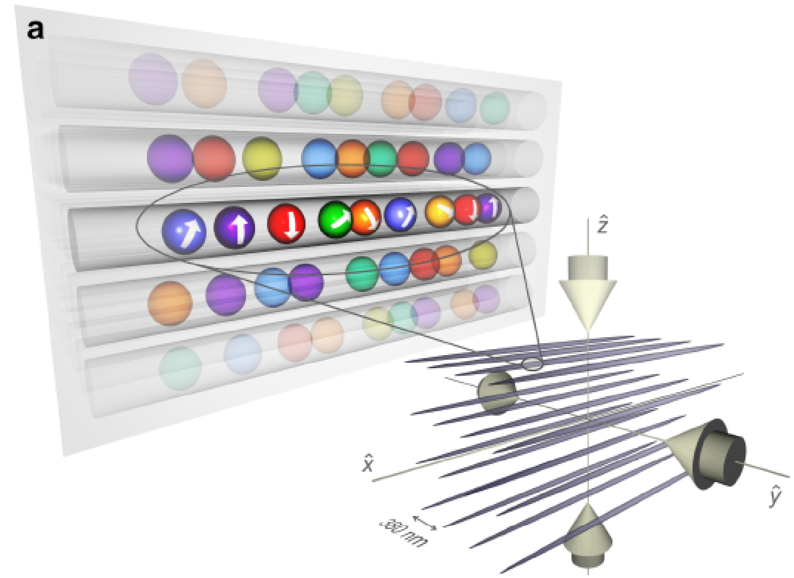
1D multi-component Fermi gases

^{173}Yb Experiments with up to r=6 components

Tight confinement – 1D regime

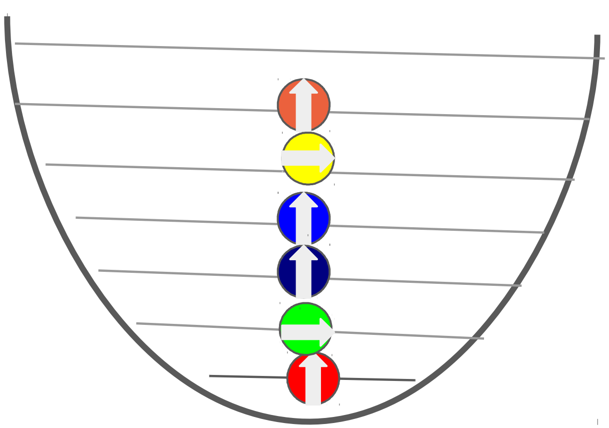
Presence of a longitudinal harmonic confinement

Repulsive interactions : g>0



[Pagano et al Nat Phys (2014)]

$$\mathcal{H} = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2} m\omega^2 x_i^2 \right] + g \sum_{i<j} \delta(x_i - x_j)$$



In the limit of strongly repulsive interactions : generalized fermionization : interactions forbid double occupancy of energy levels

Generalization of Girardeau's solutions for g → infinity

Exact solutions in the fermionized regime (I)

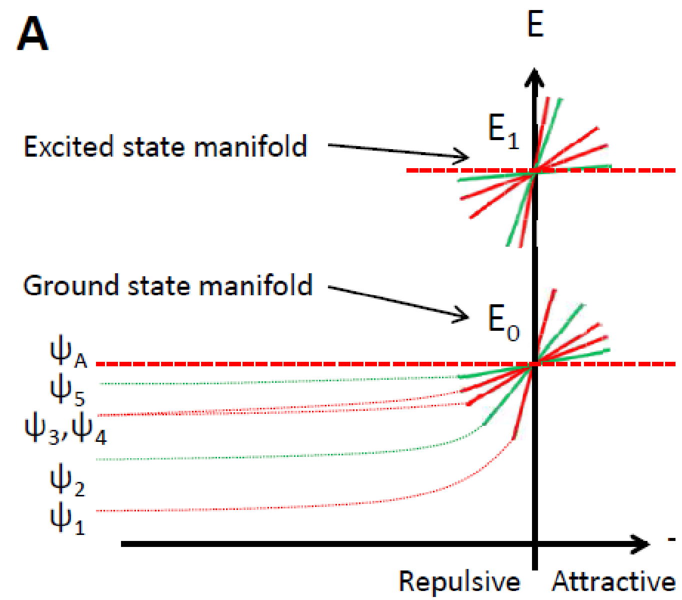
Difficulty : for a r-component Fermi gas, large degeneracy of the ground state :

$$\frac{N!}{N_1!\ldots N_r!}$$

as for multicomponent BF mixtures *[Girardeau, Minguzzi, PRL (2007)]*

Strategy : Mapping onto an ideal Fermi gas with $N=N_1+N_2+\ldots N_r$ fermions

– the ideal-Fermi gas wavefunction has the right nodes : we take *[Volosniev et al]*



[Volosniev et al Nat Phys (2015)]

indicator of a coordinate sector

$$\Psi(x_1,\ldots,x_N) = \sum_{P\in S_N} a_P \chi(x_{P(1)} < \cdots < x_{P(N)}) \Psi_A(x_1,\ldots,x_N)$$

coefficient to be determined

ideal Fermi gas wavefunction

– when exchanging two fermions belonging to the same component, the wavefunction takes a minus sign : $a_P = 1$

– one needs to fix the phase of the wavefunction when exchanging two fermions belonging to **different** components : origin of the degeneracy

Exact solution in the fermionized regime (II)

Generalization of Girardeau's wavefunction for impenetrable bosons *[Volosniev et al]*

$$\Psi(x_1,\ldots,x_N)=\sum_{P\in S_N}\boxed{a_P}\,\chi(x_{P(1)}<\cdots<x_{P(N)})\Psi_A(x_1,\ldots,x_N)$$

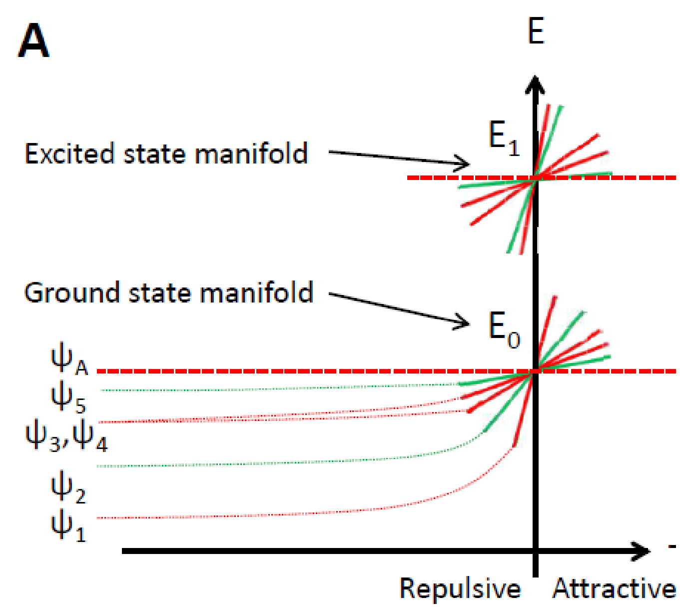
coefficients to be determined

indicator of a coordinate sector

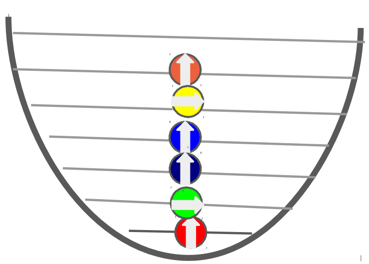
ideal Fermi gas wavefunction

The ground state wavefunction is the one which has the largest slope at decreasing interactions – related to the Tan's contact

$$K=-(m^2/\hbar^4)(\partial E/\partial g^{-1})$$



$\rightarrow$ the coefficients a_P are determined by maximizing K



I – Symmetry

The Lieb and Mattis theorem

PHYSICAL REVIEW VOLUME 125, NUMBER 1 JANUARY 1, 1962

Theory of Ferromagnetism and the Ordering of Electronic Energy Levels

Elliott Lieb and Daniel Mattis
Thomas J. Watson Research Center, International Business Machines Corporation, Yorktown Heights, New York
(Received May 25, 1961; revised manuscript received September 11, 1961)

Consider a system of N electrons in one dimension subject to an arbitrary symmetric potential, $V(x_1,\cdots,x_N)$, and let $E(S)$ be the lowest energy belonging to the total spin value S. We have proved the following theorem: $E(S)<E(S')$ if $S<S'$. Hence, the ground state is unmagnetized. The theorem also holds

Two component fermions (electrons) : the ground state has the smallest possible spin compatible with the fermion imbalance

Example with two fermions :

$$ {}_0{}^0\Psi = \Phi_{Sy}(\mathbf{r}_1,\mathbf{r}_2)[(+-)-(-+)], $$

The spin part has S=0 and is antisymmetric. The spatial part is symmetric. (→ The total wavefunction is antisymmetric)

Absence of ferromagnetism for any finite interactions

see also *[Barth and Zerger Ann. Phys. 326, 2544, 2011]*

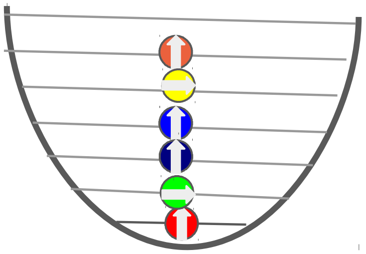
Questions :

Magnetic structure for systems with more than two spin components :

– How to characterize it ?

– How to observe it ?

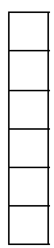
→ not an ensemble of spin ½ particles,
each component corresponds to a ‘color’



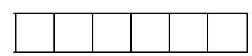
Symmetry characterization for multicomponent gases

The Young tableaux indicate the symmetry under exchange of particles belonging to each component

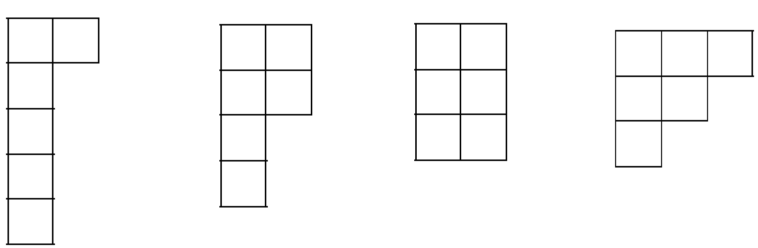
Examples for 6 fermions :



Fully antisymmetric spatial wavefunction



Fully symmetric spatial wavefunction



Intermediate symmetry : antisymmetric wrt columns and symmetric wrt rows

How to associate Young tableaux to wavefunctions

Use the class-sum operators *[Katriel, J. Phys. A, 26, 135 (1993)]*

$$\Gamma^{(k)} = \sum_{i_1 < \dots i_k} (i_1 \dots i_k)$$

cyclic permutation
of k elements

For the transposition class $\Gamma^{(2)}$ its eigenvalue γ_2 allows to link to the Young tableau according to

$$\gamma_2 = \frac{1}{2} \sum_i \lambda_i (\lambda_i - 2i + 1)$$

line of Young tableau

number of boxes
in the Young tableau

[J. Decamp et al, NJP 18, 055011 (2016)]

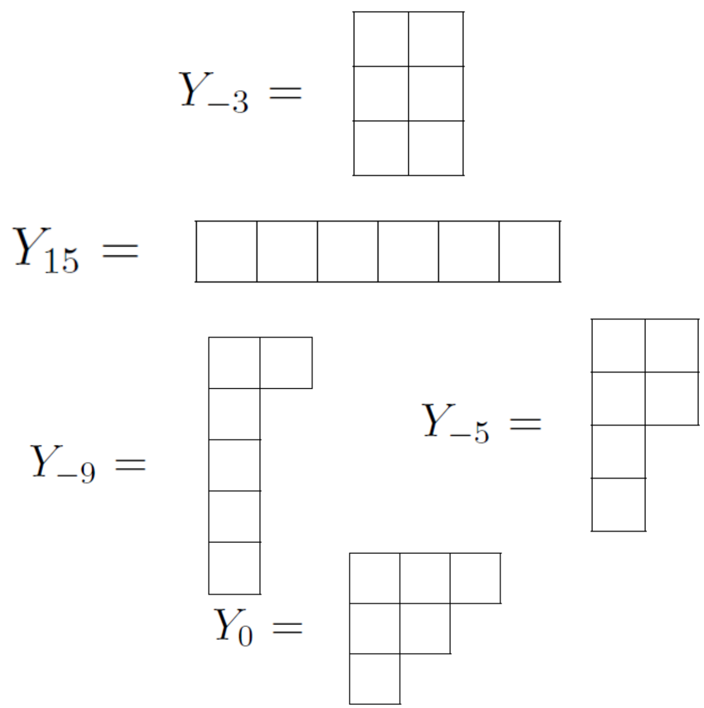
Symmetry of the wavefunctions : results

Take total N=6 fermions, various combinations among the components

The ground state spatial wavefunction has a single Young tableau → a definite symmetry

symmetry

System	$A_{\max}$
$r = 2,\ N_1 = N_2 = 3$	Y_{-3}
$r = 3,\ N_1 = N_2 = N_3 = 2$	Y_3
$r = 6,\ N_1 = \ldots = N_6 = 1$	Y_{15}
$r = 2,\ N_1 = 5,\ N_2 = 1$	Y_{-9}
$r = 2,\ N_1 = 4,\ N_2 = 2$	Y_{-5}
$r = 3,\ N_1 = 3,\ N_2 = 2,\ N_3 = 1$	Y_0



Y_γ is the Young tableau with eigenvalue γ of the transposition class-sum operator $\Gamma^{(2)}$

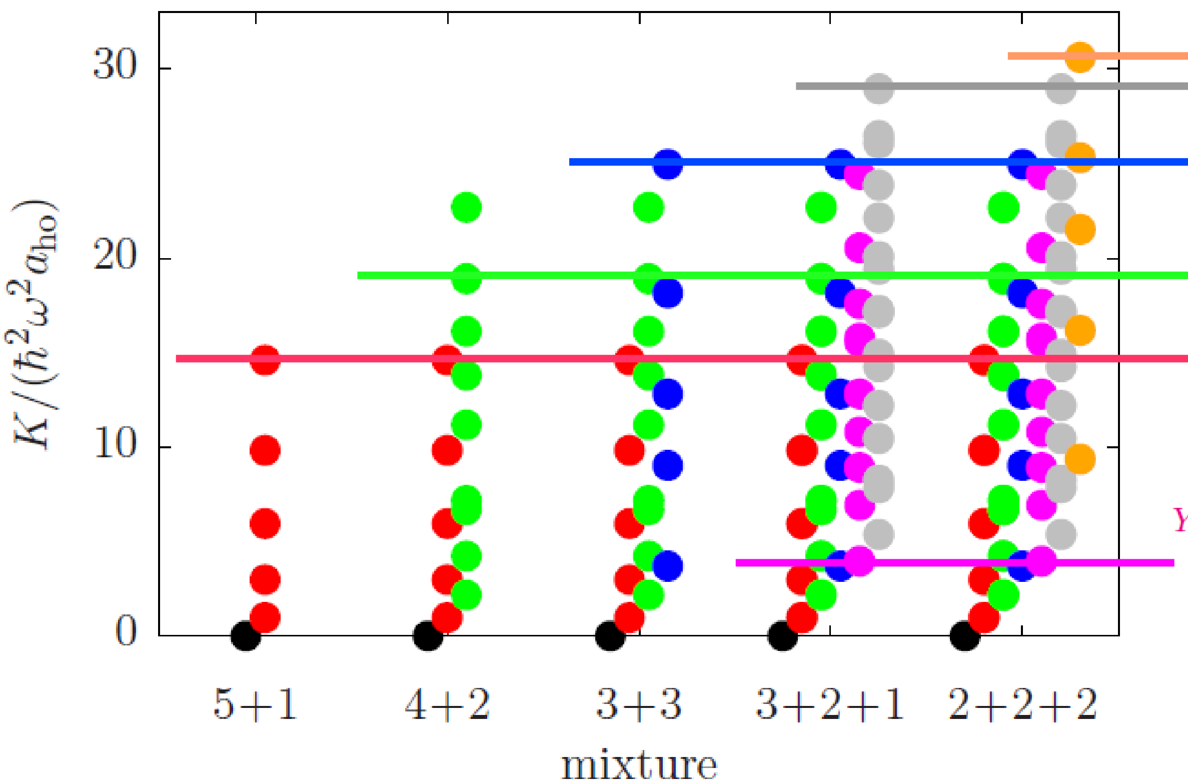
The ground-state configuration is the most symmetric one compatible with imbalance : Generalization of the Lieb-Mattis theorem to multicomponent Fermi gases

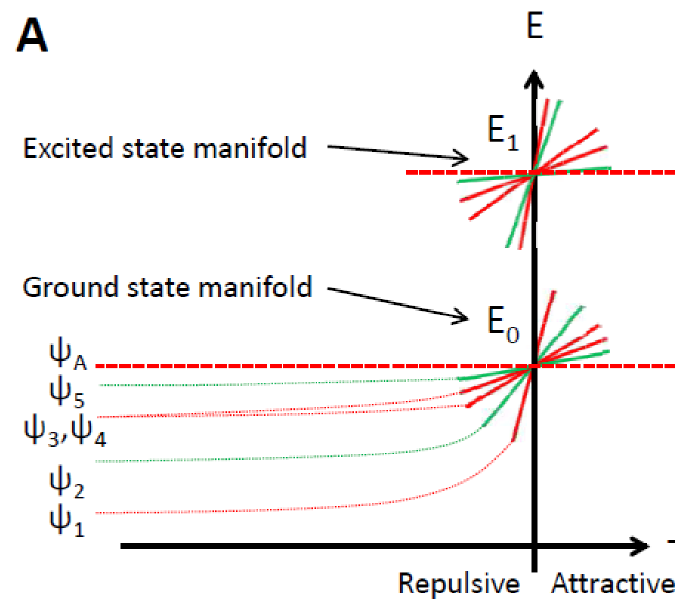
[J. Decamp et al, NJP 18, 055011 (2016)]

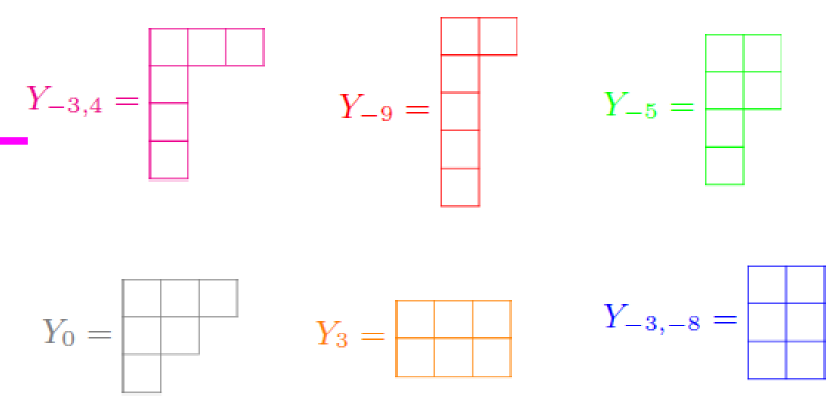
Total interaction parameter and symmetry

N=6 fermions, slope of the energy curves

$$K = -\partial E/\partial g^{-}1|_{g\rightarrow\infty}$$







Symmetry spectroscopy : a unique value for K for a given symmetry

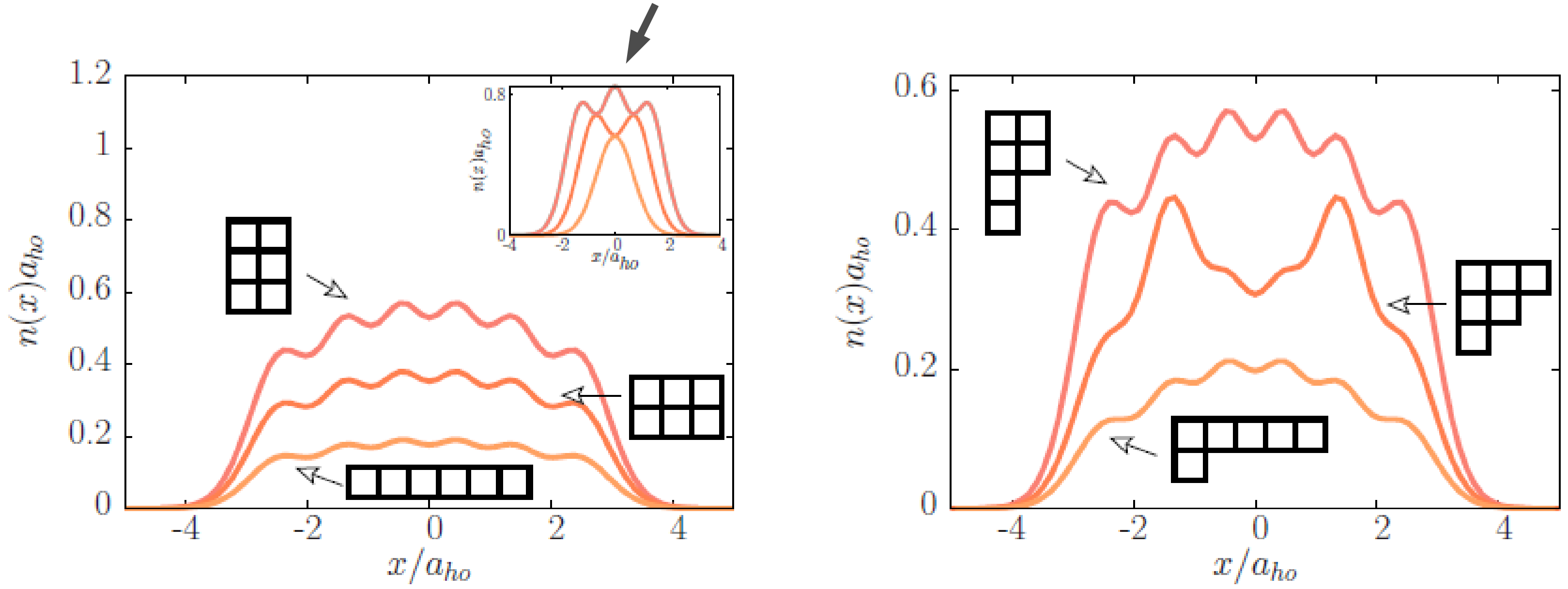
[Decamp et al, PRA 94, 053614 (2016)]

II – Density profiles

Density profiles and symmetry for a strongly correlated Fermi gas : ground and excited states

N=6 fermions, symmetric mixtures 1+1+1+1+1+1, 2+2+2, 3+3

noninteracting profiles



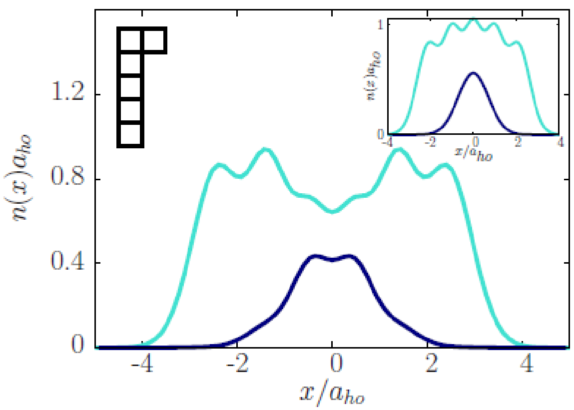
The density profiles depend on the symmetry of the mixture

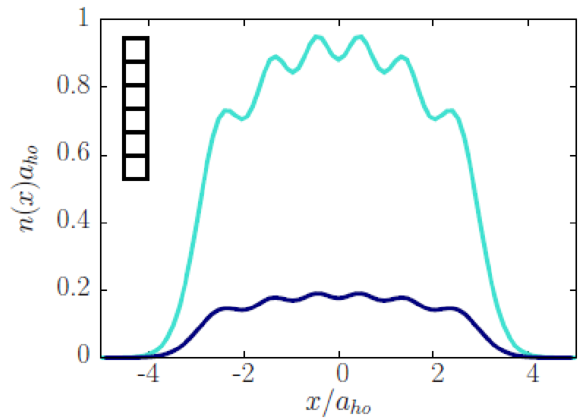
The higher excited states are less and less symmetric than the ground state : highest excited state – 'ferromagnetic'

[J. Decamp et al, NJP 18, 055011 (2016)]

Density profiles and symmetry for a strongly correlated Fermi gas : ground and excited states

N=6 fermions, imbalanced mixtures 5+1



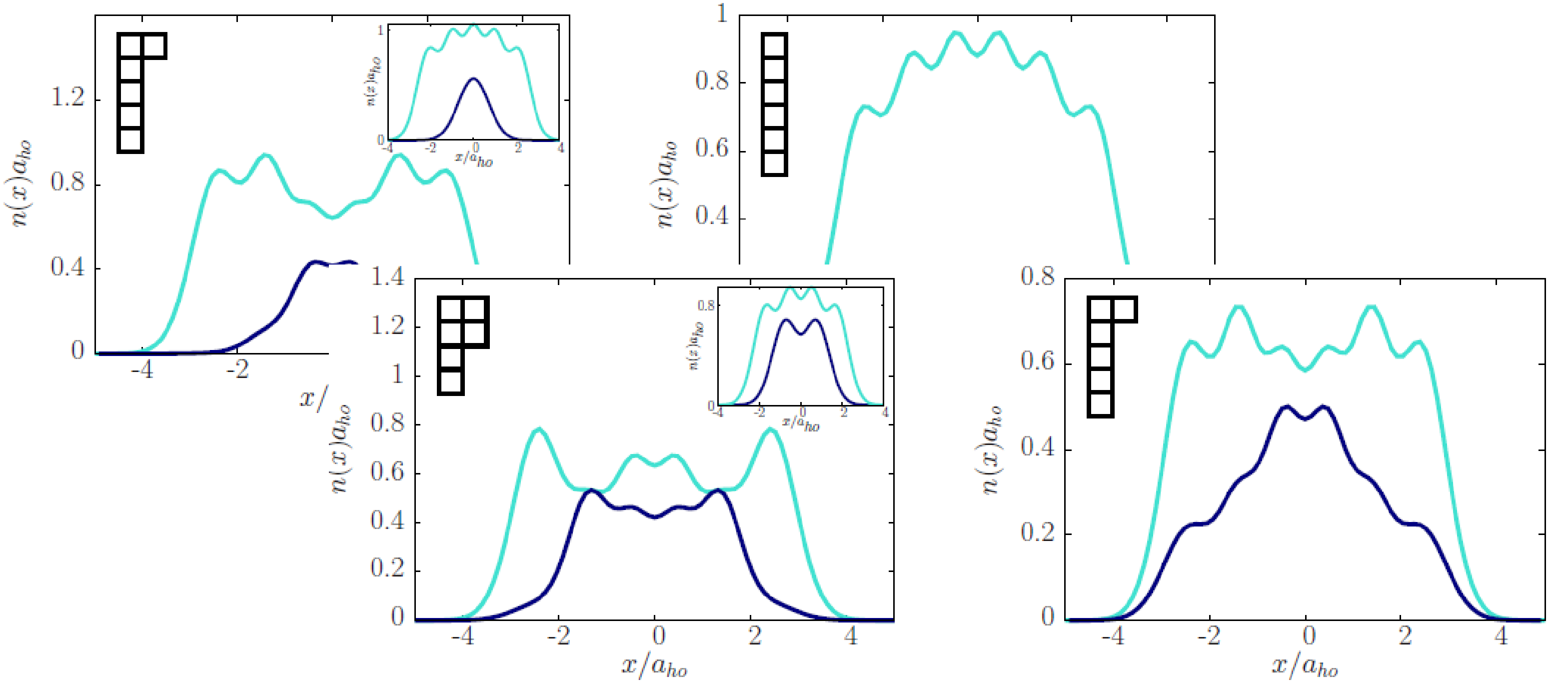


Repulsive interactions :
hole in the majority distribution, polaron

The excited state is fully antiymmetric : the density profile coincides with the one of a noninteracting Fermi gas with N=6

Density profiles and symmetry for a strongly correlated Fermi gas : ground and excited states

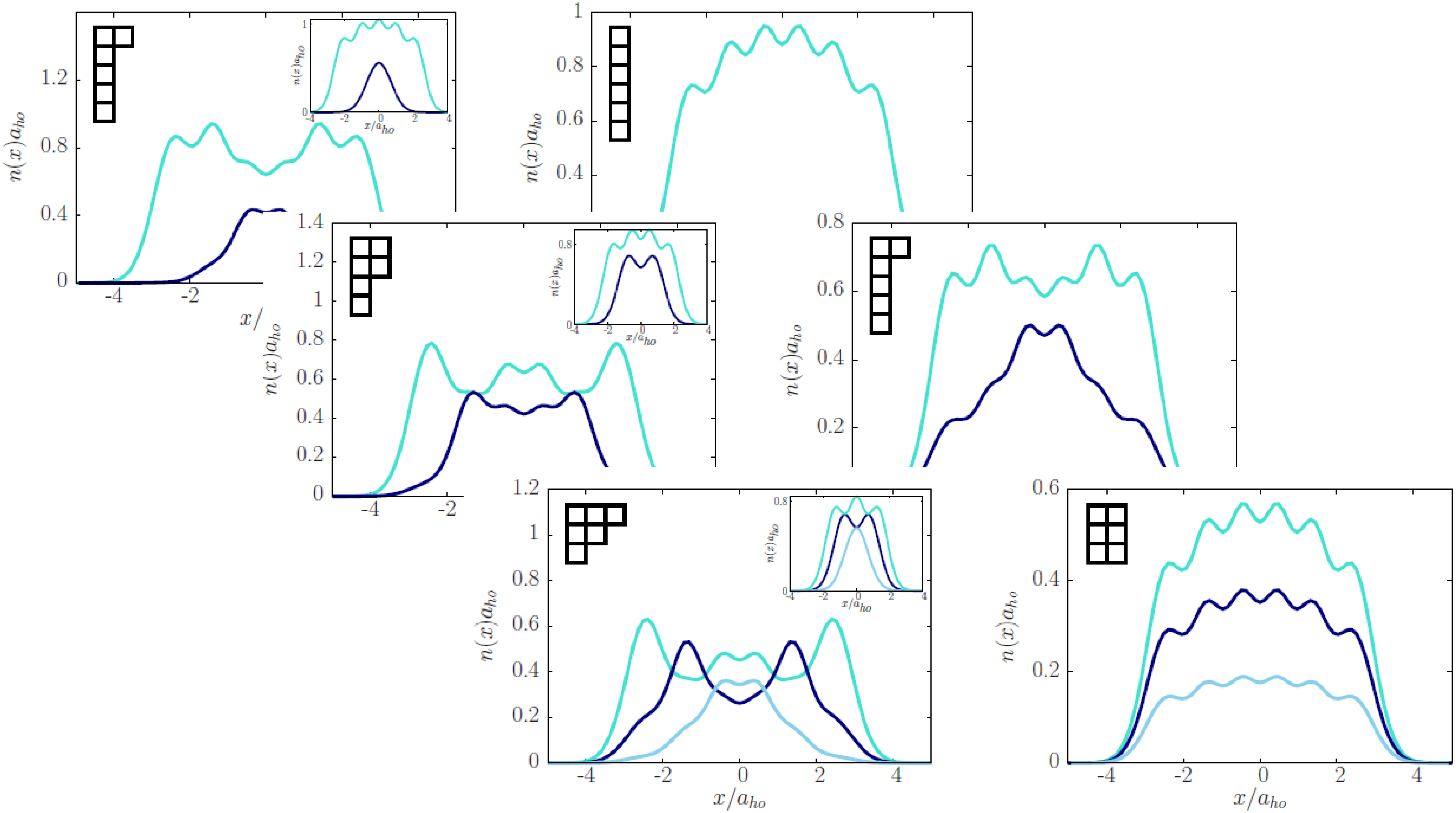
N=6 fermions, imbalanced mixtures 5+1, 4+2



Alternance of the two components: antiferromagnet

Density profiles and symmetry for a strongly correlated Fermi gas : ground and excited states

N=6 fermions, imbalanced mixtures 5+1, 4+2, 3+2+1

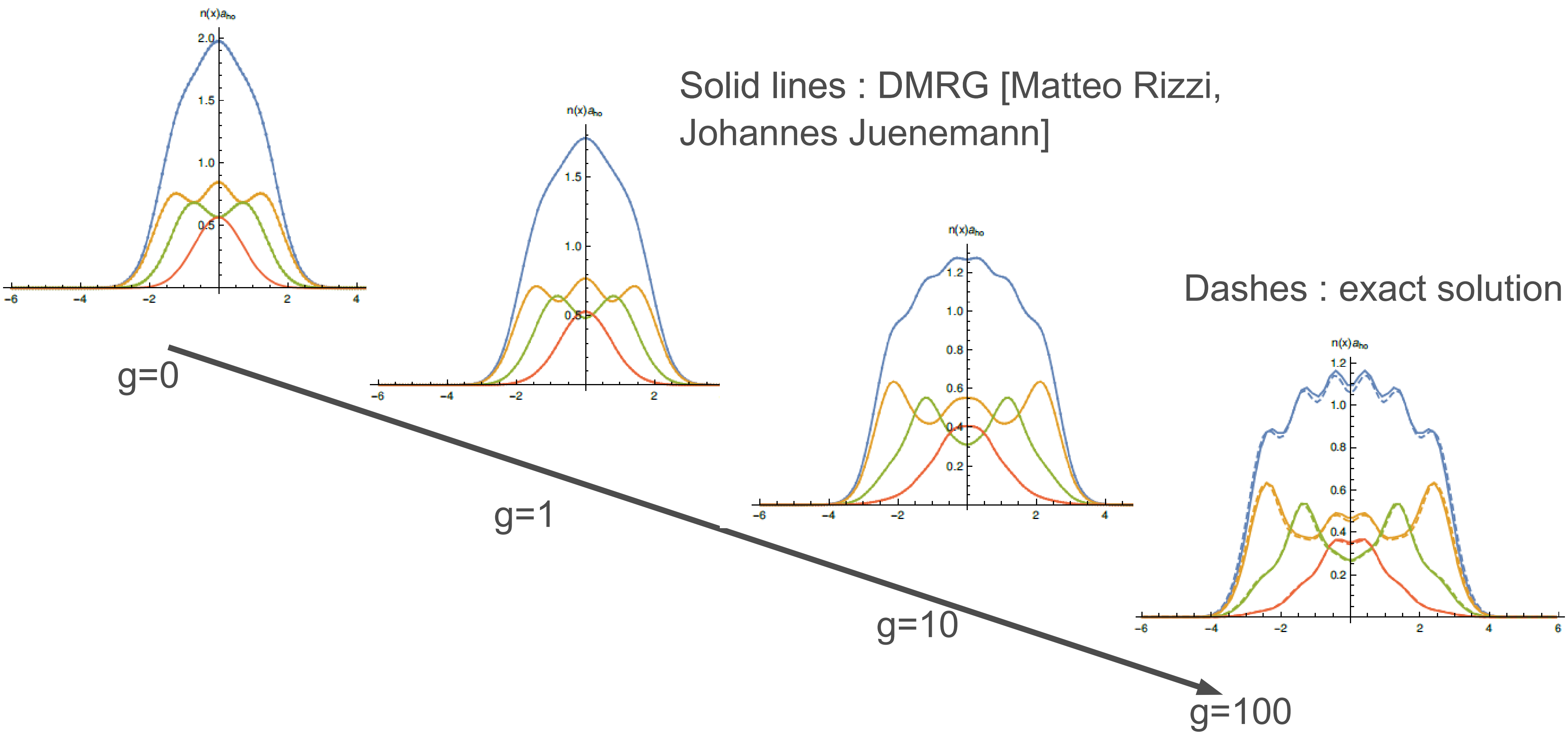


Link between symmetry and spatial shape

[J. Decamp et al, NJP 18, 055011 (2016)]

How strong the interactions should be to see correlation effects?

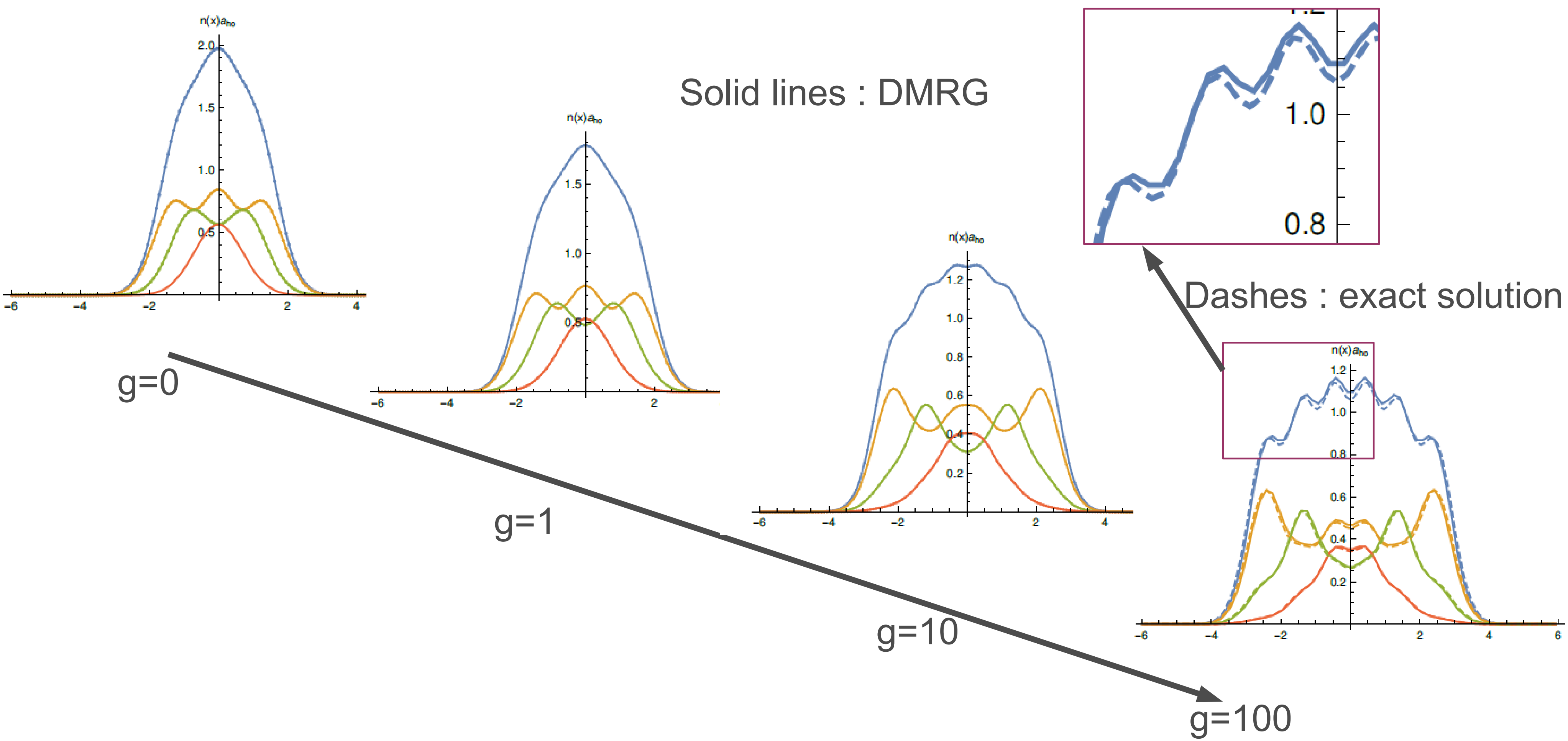
Analysis at finite interactions, N= 3+2+1



(g in harmonic oscillator units)

How strong the interactions should be to see correlation effects?

Analysis at finite interactions, N= 3+2+1

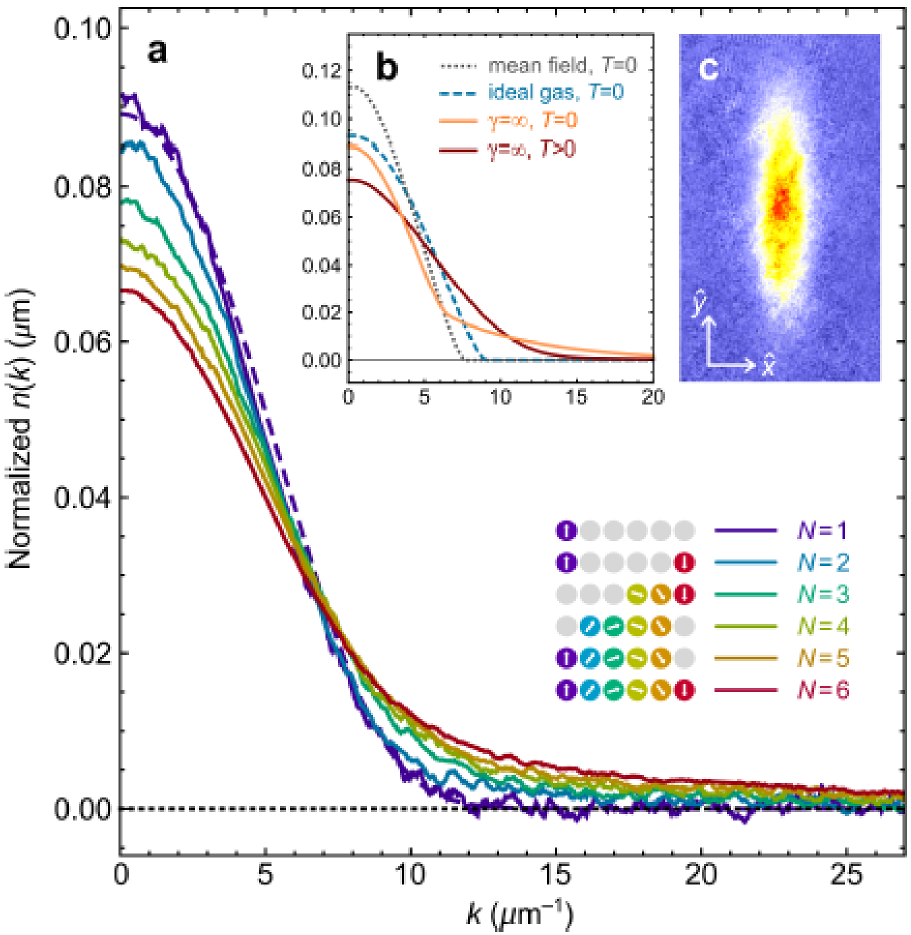


(g in harmonic oscillator units)

III – Momentum distributions

Momentum distributions for multicomponent fermions

Accurately measured in experiments



[Pagano et al Nat Phys (2014)]

Effect of confinement ?

Effect of interactions ?

Effect of number of components ?

Effects of temperature ?

Momentum distributions for multicomponent fermions

Definition

Density in momentum space, Fourier transform of the one body density matrix

$$\rho_\nu(x_1, x_1') = N_\nu \int \mathrm{d}x_2 \ldots \mathrm{d}x_N \Psi(X)\Psi(X')$$

where $X = (x_1, \ldots, x_N)$ $X' = (x_1', x_2, \ldots, x_N)$

and the first coordinate belongs to the component ν.

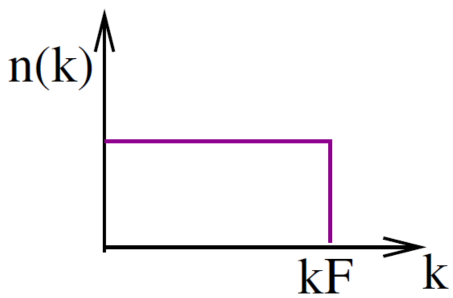
Momentum distribution for the fermionic component ν :

$$n_\nu(k) = \iint \mathrm{d}x\mathrm{d}y \rho_\nu(x, y) e^{-ik(x-y)}$$

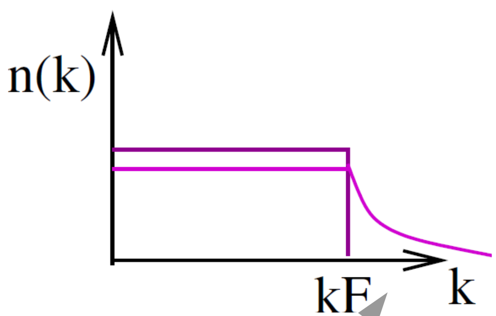
Valid for arbitrary interactions and external confinement

Momentum distribution of a Fermi gas

Basic facts – homogeneous system results



Noninteracting fermions, homogeneous system : a sharp Fermi edge at k=kF



Tails : effect of interactions

Multicomponent interacting 1D fermions, homogeneous system :

– a power-law discontinuity at k=kF from Luttinger liquid / conformal field theory *[Frahm, et al (1993)]*

$$n_\nu(k) \sim |k - k_F|^\alpha$$

– large momentum tails with universal power law (beyond Luttinger-liquid theory) *[Barth et al (2011)]*

$$n_\nu(k) \ \sim \ \mathcal{C}_\nu k^{-4}$$

Large-momentum tails of the momentum distribution

$n_\nu(k) \sim \mathcal{C}_\nu k^{-4}$ Power-law tails : due to the behaviour of the many-body wavefunction at short distances, fixed by the contact interactions

$$\partial_x \Psi(0^+) - \partial_x \Psi(0^-) = (2mg/\hbar^2)\Psi(0)$$

The weight of the tails (Tan's contact) is related to the two-body correlation function

$$\mathcal{C}_{tot}^{dens} = \frac{n^2}{\pi a_1^2}\frac{r-1}{r} g_{12}^{(2)}(0,0)$$

$$a_1 = -\frac{1}{m_r g}$$

Tan's relations : also related to the interaction energy of the specie ν with all the other species

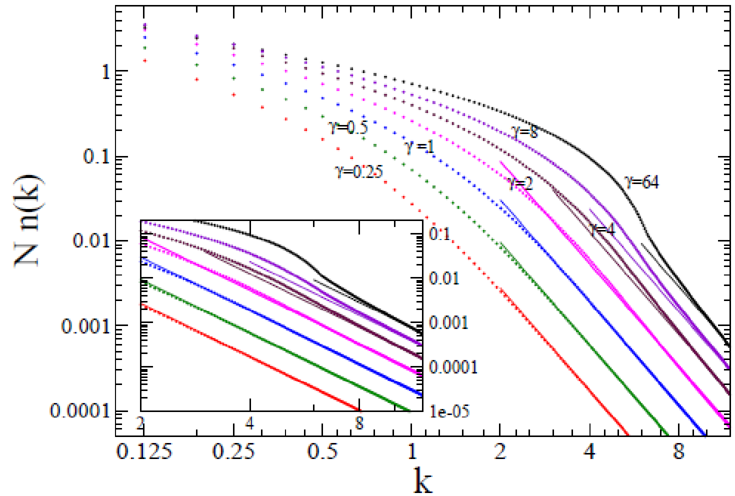
$$g\left\langle H_{int,\nu}\right\rangle = 2\pi\mathcal{C}_\nu$$

Can be obtained from the ground state energy using the Hellmann-Feynman theorem

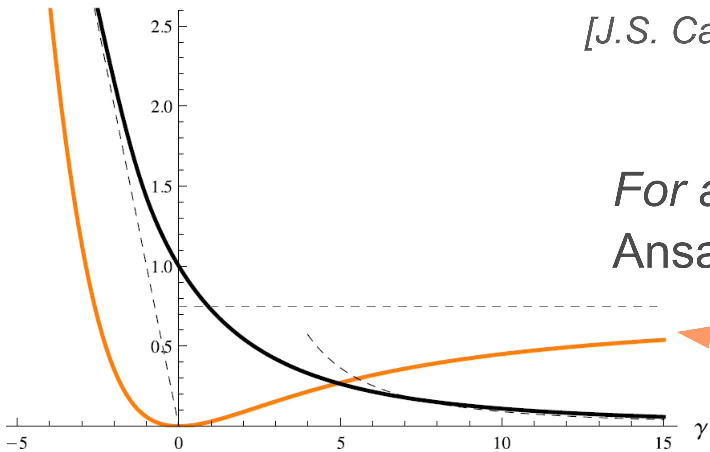
Large-momentum tails for a homogeneous gas

$$n_\nu(k) \sim \mathcal{C}_\nu k^{-4}$$ The tails increase with interaction strength

For a Bose gas, from Bethe Ansatz :



[J.S. Caux, P. Calabrese, N.A. Slavnov, (2007)]



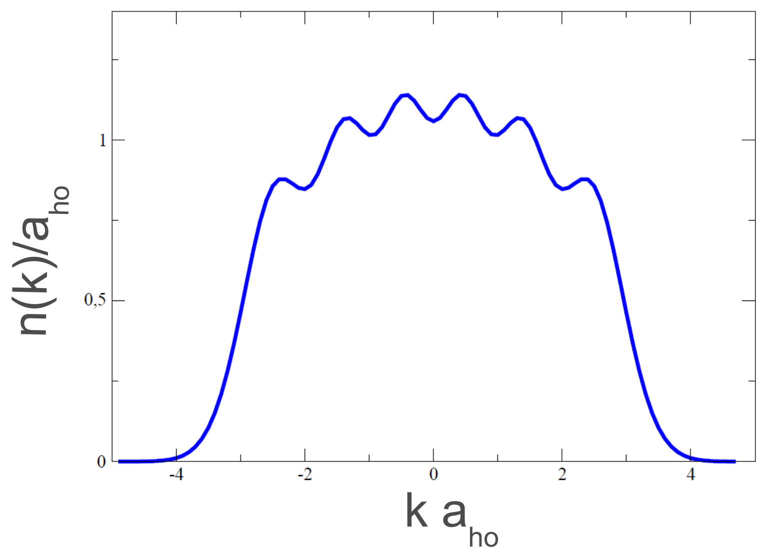
[M. Barth and W. Zwerger, (2011)]

For a two-component Fermi gas, from the Bethe Ansatz equation of state :

Weight of the momentum distribution tails

Two-body correlation function

Momentum distribution for noninteracting fermions in harmonic trap



Noninteracting fermions, same as density profile due to the x – p duality of the harmonic oscillator Hamiltonian

Number of peaks = number of fermions

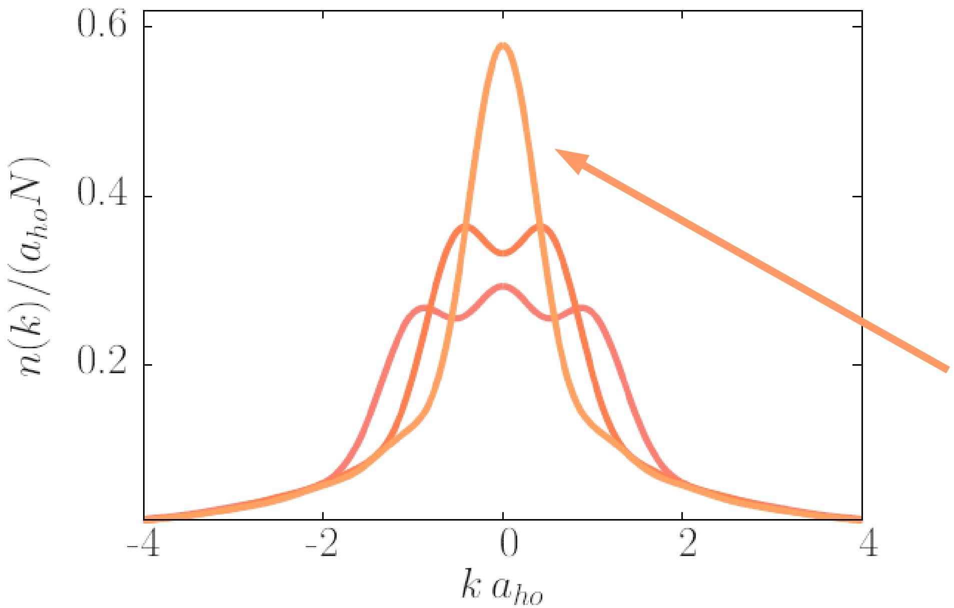
Oscillations in the density profiles :

~ Friedel oscillations

~ 1/N decay

Momentum distributions for a multicomponent Fermi gas at infinitely strong interactions in harmonic trap

N=6 fermions, symmetric mixtures 1+1+1+1+1+1, 2+2+2, 3+3

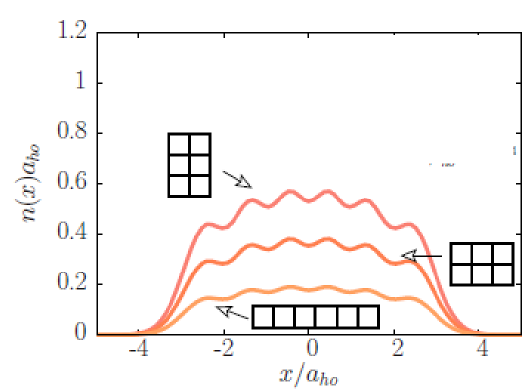


From the exact solution

Number of peaks = number of fermions in each component *[Deuretzbacher et al, arXiv:1602.0681]*

The case 1+1+1+1+1+1 has the same momentum distribution as a bosonic Tonks-Girardeau gas with N_B=6

Corresponding density profiles :



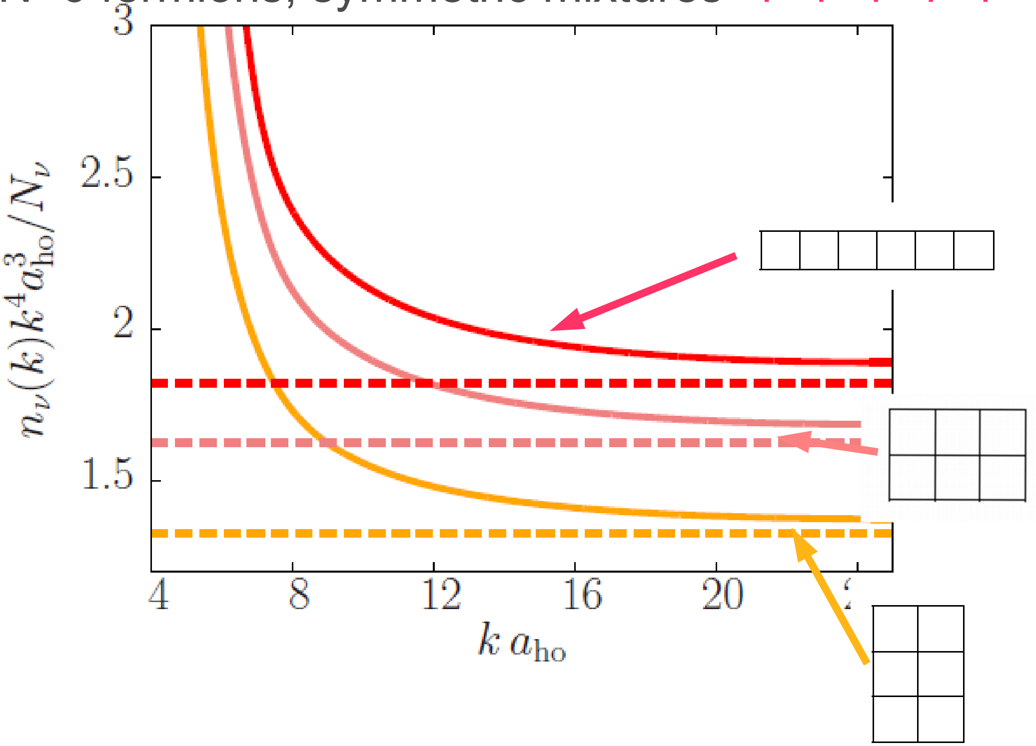
A strong effect of interactions :

- ***reduction of the width of the zero-momentum peak / opposite to broadening of the density profiles***
- ***large momentum tails***

[Decamp et al, PRA 94, 053614 (2016)]

High-momentum tails for a multicomponent Fermi gas at infinitely strong interactions in harmonic trap

N=6 fermions, symmetric mixtures 1+1+1+1+1+1, 2+2+2, 3+3



From the exact solution for n(k) (solid lines)

Asymptotic behaviour from the 1/g expansion of the energy (dashed lines)

The most symmetric wavefunction has the largest tails in n(k)

Symmetry of the mixture from the tails of the momentum distribution !

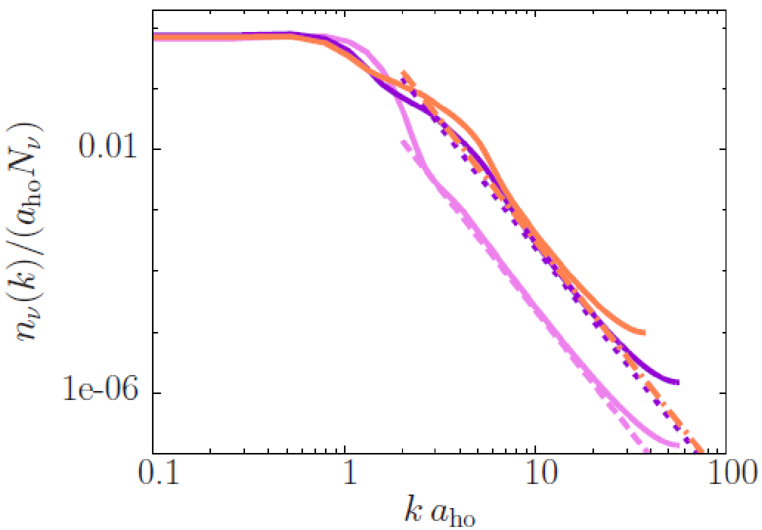
A way to probe (generalized) antiferromagnetism

[Decamp et al, PRA 94, 053614 (2016)]

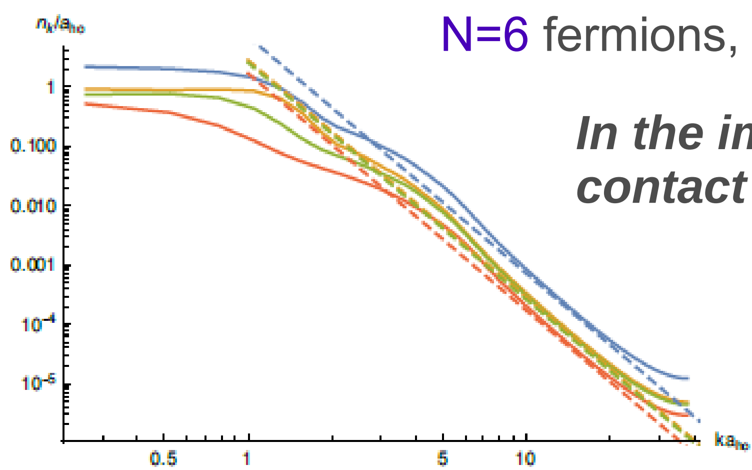
High-momentum tails for a multicomponent Fermi gas at finite interactions, in harmonic trap

Numerical calculations with DMRG

N=6 fermions, log scale, mixture 3+3 g=1, 10, 100



The tails increase with interaction strength



N=6 fermions, log scale, mixture 3+2+1 g=10

In the imbalanced case, there is a different contact for each component

[Decamp et al, PRA 94, 053614 (2016)]

Local-density approximation for the momentum distribution tails of a 1D interacting Fermi gas in harmonic trap

Based on the exact equation of state from Bethe Ansatz *[X.W. Guan et al PRA 2012]*

$$E/L = (\hbar^2/2m)\rho^3 e(\gamma)$$

$$\gamma = m g_{1\mathrm{D}}/\hbar^2 \rho$$

Inhomogeneous density profile – from minimization of energy functional :

$$\frac{3}{2}\frac{\hbar^2}{m}\rho^2 e(\gamma) - \frac{1}{2} g_{1\mathrm{D}} \rho e'(\gamma) = \mu - V_{\mathrm{ext}}(x)$$

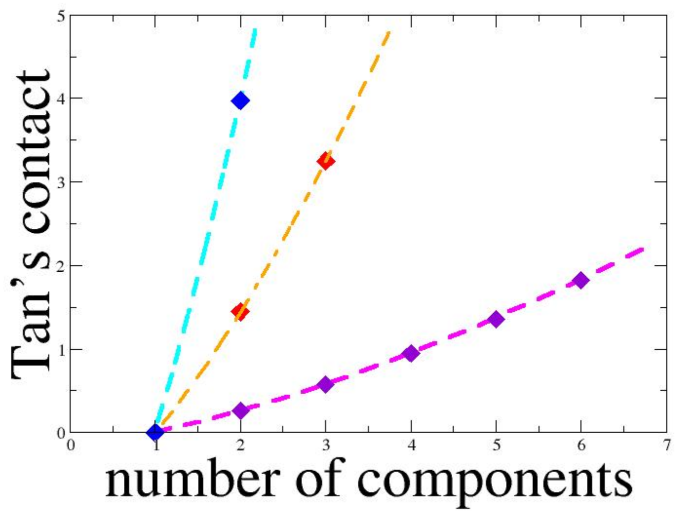
Tan’s contact for the inhomogeneous Fermi gas :

$$\mathcal{C}_{\mathrm{tot}} = \frac{g_{1\mathrm{D}}^2}{2\pi} \int dx \rho^2(x) e'(\gamma)$$

[Decamp et al, PRA 94, 053614 (2016)]

Contact vs number of components at infinitely strong interactions

r-component Fermi gas in harmonic trap, zero temperature $N_\nu = N/r$



Exact calculations in the trap N_ν=1,2,3

LDA on Bethe-Ansatz equation of state

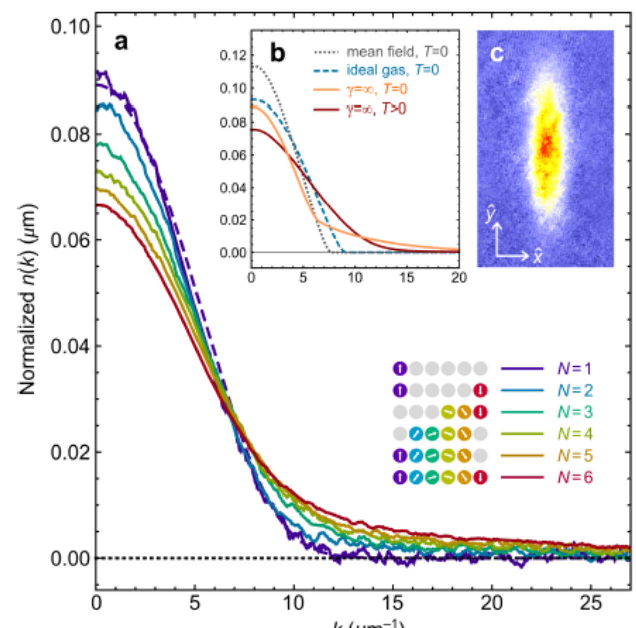
[X.W. Guan et al PRA 2012]

$$C_\nu(\infty) = \frac{128\sqrt{2} Z_1(r) N^{5/2}}{45 r \pi^3}$$

Related to digamma function ψ

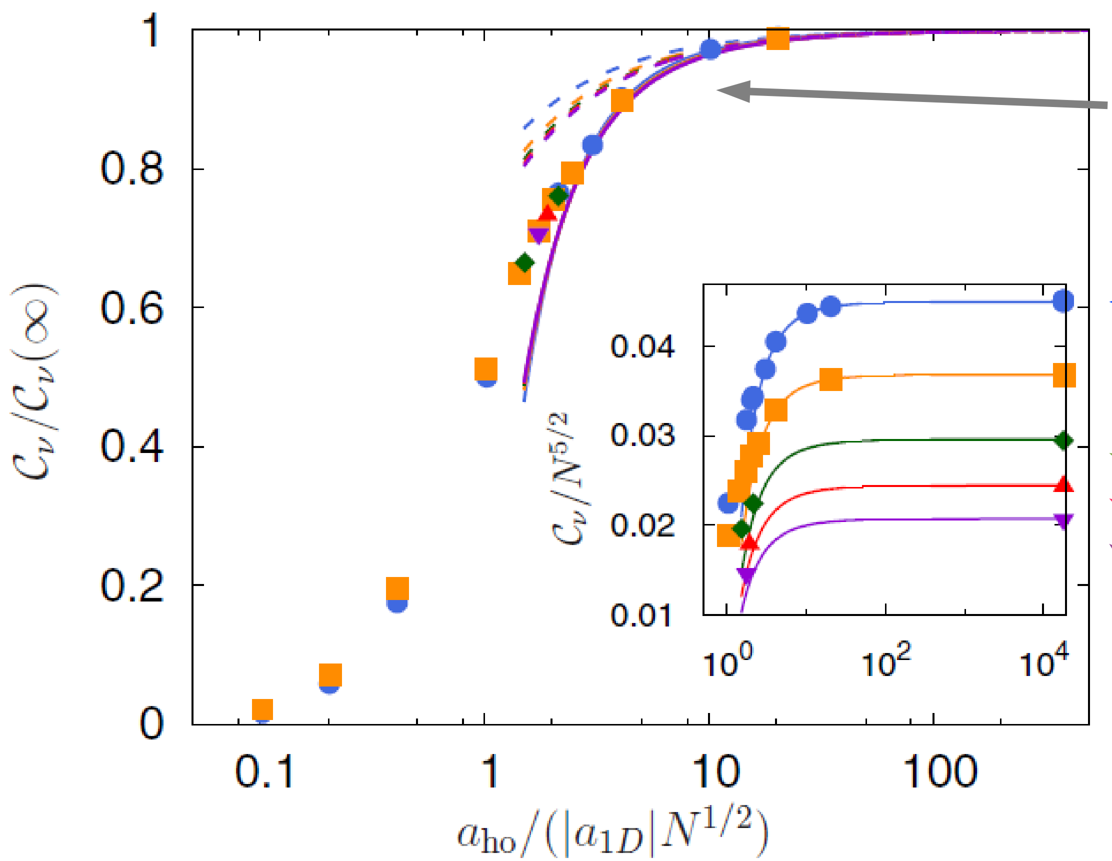
The tails increase with increasing number of components at fixed N_ν

– also in the Florence experiment !!



Contact vs interactions : DMRG + LDA results

fermionic mixture in harmonic trap at various numbers of fermions, zero temperature



Lines : LDA on the strong-coupling expansion of the BA equation of state to order 1 (dash) and 2 (solid)

Two components

Three components

Four components

Five components

Six components

Strong correlations = = large tails of the momentum distribution

Scaling : the numerical data collapse once dividing by $\mathcal{C}_\nu(\infty)$

[Decamp et al, PRA 94, 053614 (2016)]

High-momentum tails for a Fermi gas at high temperature

Generalization of the Tan's sweep theorem at finite temperature :

$$\left(\frac{\mathrm{d}\Delta\Omega_\nu}{\mathrm{d}a_{1D}}\right)_{\mu,T} = \frac{\pi\hbar^2}{m}\mathcal{C}_\nu$$

$$\Omega = \Omega^{(1)} + \frac{1}{2}\sum_\nu \Delta\Omega_\nu$$ grand-thermodynamic potential, obtained by summing over all the components

High-temperature regime : we use a *virial approach*

– virial expansion for the grand-thermodynamic potential : $\Delta\Omega_\nu = -2k_BT\left(Q_2 - \frac{Q_1^2}{2}\right) z_\nu \sum_{\mu\neq\nu} z_\mu$

$$\mathcal{C}_\nu = \frac{4Q_1}{\Lambda_{dB}^3} c_2 z_\nu \sum_{\mu\neq\nu} z_\mu$$

with $c_2 = -\frac{\partial(Q_2/Q_1)}{\partial(a_{1D}/\Lambda_{dB})}$

– solution for the two-body problem in harmonic trap *[Th. Busch et al, Found. Phys. 28, 549 (1998)]*

$$Q_2 \;=\; Q_1 \sum\nolimits_\kappa e^{-\epsilon_\kappa^{rel}/k_BT} \qquad \epsilon_\kappa^{rel} = \hbar\omega(\kappa + 1/2) \qquad \frac{\Gamma(-\kappa/2)}{\Gamma(-\kappa/2+1/2)} = \frac{\sqrt{2}a_{1D}}{a_{HO}}$$

High-momentum tails at finite (high) temperature

High-temperature regime, infinite interactions $a_{1D} \to 0$

– **Universality : no energy or length scale associated to interactions**

the virial coefficient for the contact is a number – does not depend on interaction or temperature *[P. Vignolo, A. Minguzzi, PRL 2013]*

$$c_2 = -\frac{\partial(Q_2/Q_1)}{\partial(a_{1D}/\Lambda_{dB})}$$

$$c_2 = 1/\sqrt{2}$$

High-momentum tails at finite (high) temperature

High-temperature regime, infinite interactions

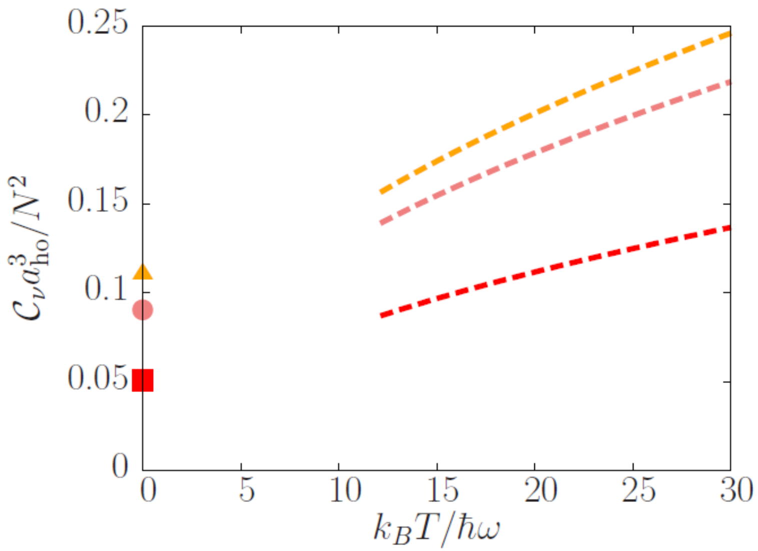
– **Universality : no energy or length scale associated to interactions**

the virial coefficient for the contact is a number – does not depend on interaction or temperature *[P. Vignolo, A. Minguzzi, PRL 2013]*

$$c_2 = 1/\sqrt{2}$$

N=6 fermions, symmetric mixtures

1+1+1+1+1+1, 2+2+2, 3+3



[Decamp et al, PRA 94, 053614 (2016)]

– High-temperature contact coefficients :

$$\mathcal{C}_\nu = \frac{1}{(\sqrt{\pi}a_{HO})^3}\sqrt{\frac{k_BT}{\hbar\omega}}N_\nu\sum_{\mu\neq\nu}N_\mu$$

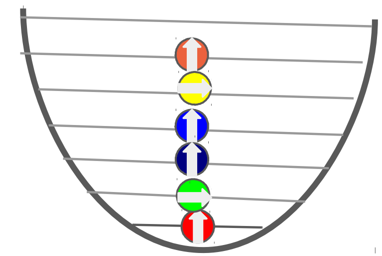
The tails of the momentum distribution increase with temperature

Conclusions

1D multicomponent fermions with strong repulsive interactions

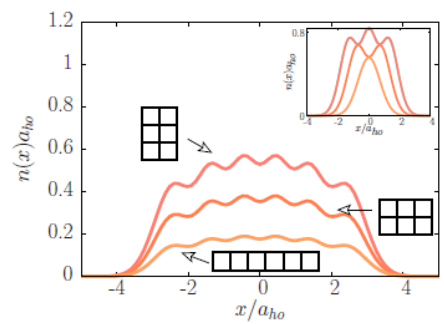
– Exact analytical solution at infinite interactions,

– Numerical DMRG results at arbitrary interactions

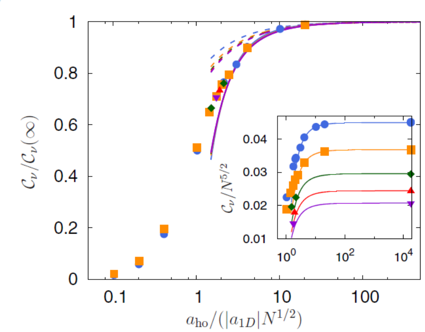


The ground state has the most symmetric wavefunction

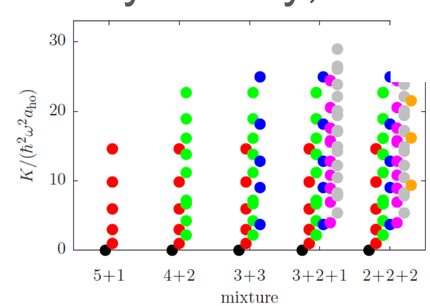
Density profiles for different symmetries are different



Momentum distribution tails increase with interaction strength, number of components and temperature



Symmetry spectroscopy : tails uniquely associated to a symmetry, largest tails for the most symmetric configuration



Outlook

1D multicomponent fermions with strong repulsive interactions :

– Larger N

– Luttinger liquid theory & beyond

– Dynamical properties

Momentum distributions at finite temperature

A big thanks to...

Pacome Armagnat (CEA, Grenoble)

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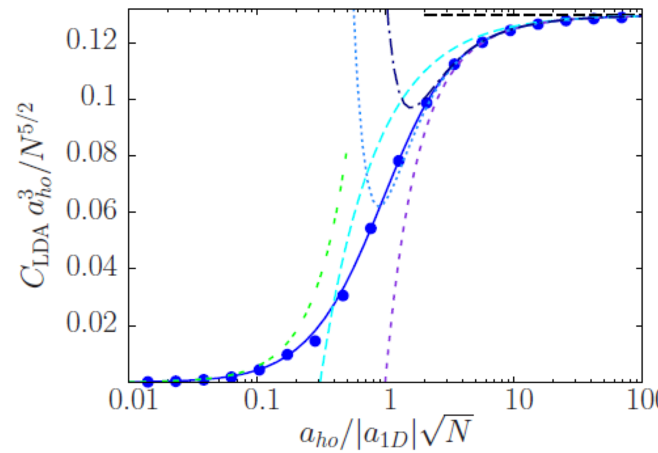
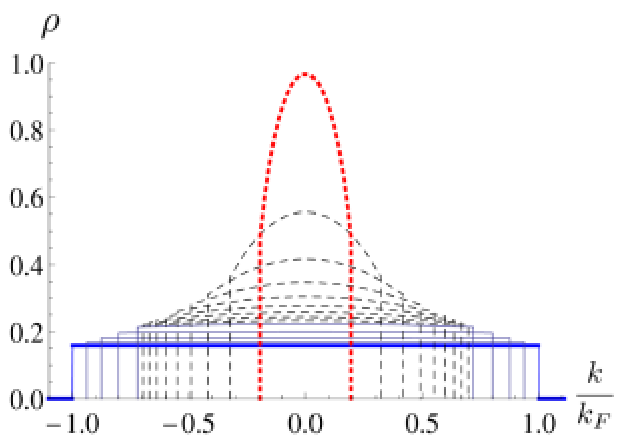
Matteo Rizzi (JGU, Mainz)

Other Grenoble results

Tan's contact of a harmonically trapped one-dimensional Bose gas: strong-coupling expansion and conjectural approach at arbitrary interactions *[EPJ – ST 226, 1583 (2017)]*





Ground-state energy and excitation spectrum of the Lieb-Liniger model : accurate analytical results and conjectures about the exact solution *[SciPost Phys. 3, 003 (2017)]*

$$\rho_1(x;\, x') = \frac{1}{L}\sum_{l=0}^{+\infty} c_l (n|x-x'|)^l$$

A connection between non-local one-body and local three-body correlations of the Lieb-Liniger model *[arXiv:1705.02100]*

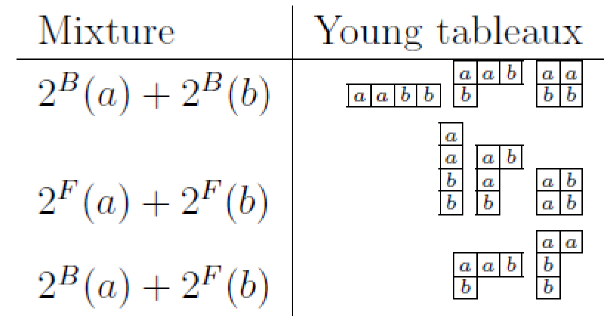
$$\rho_k(x_1, \ldots x_k; x'_1, \ldots, x'_k) \equiv \int dx_{k+1} \ldots dx_N$$
$$\chi_N^*(x'_1, \ldots, x'_k, x_{k+1}, \ldots, x_N)\chi_N(x_1, \ldots, x_N)$$

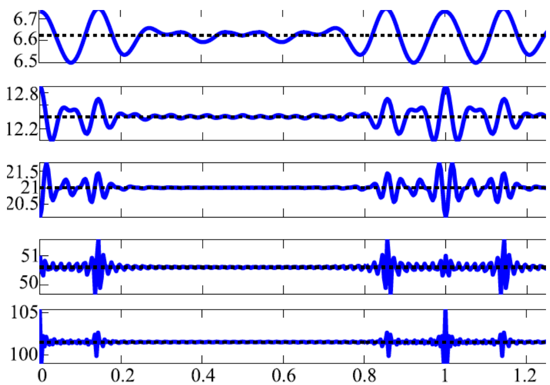
$$24c_4 - 2\gamma^2 g_3 = e_4 - \gamma e'_4$$

$$g_k \equiv \frac{N!}{(N-k)!}\frac{\rho_k(0, \ldots, 0; 0, \ldots, 0)}{n^k}$$

Other Grenoble results

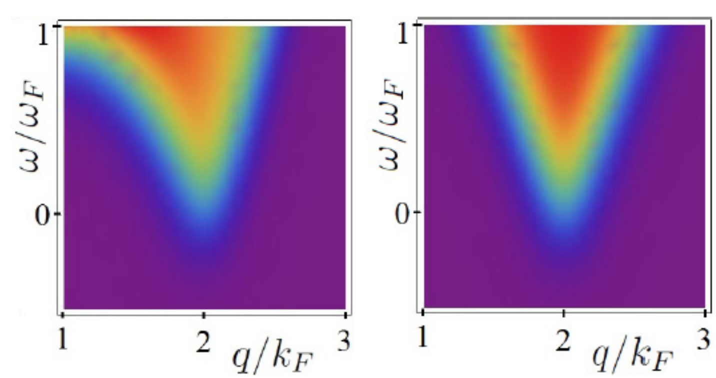
Strongly correlated one-dimensional Bose-Fermi quantum mixtures: symmetry and correlations *[arXiv:1707.09206]*





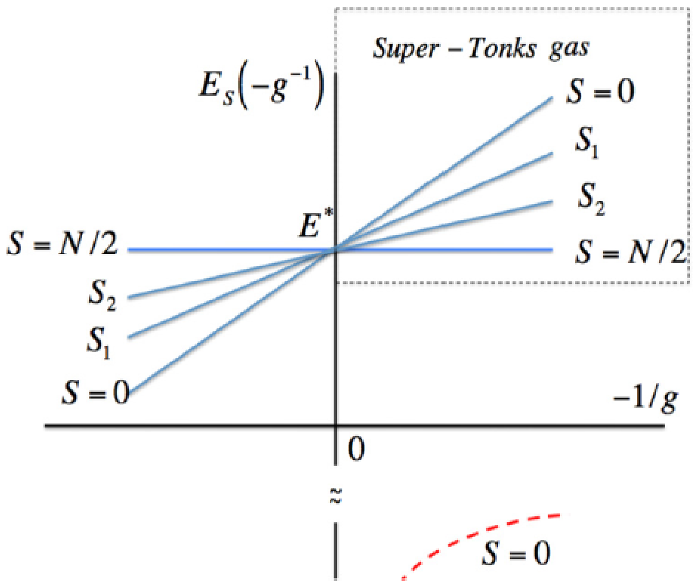
Dynamical depinning of a Tonks-Girardeau gas *[Phys Rev A, 92, 063605]*

Dynamic structure factor and drag force in a strongly interacting 1D Bose gas at finite temperature *[Phys Rev A 91 063619 (2015)]*



Can one have ferromagnetism then ?

The highest excited branch at infinite interactions has the largest spin



[Cui and Ho, PRA 89, 023611 (2014)]

Ferromagnetism possible in the lowest gas state of the system with large attractive interactions *(super-Tonks regime)*